Quotient Spaces, Differential Forms, and de Rham Cohomology

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1 Closedness & Exactness

Recall from chapter 5.8 that there are two important and related properties differential forms (or vector fields) can have. First, we can ask whether a form/field $\mathbf{G} = G_1 dx_1 + \ldots + G_n dx_n$ has a potential: if there is some function f such that $\mathbf{G} = \nabla f$. A $\mathbf{G}$ which has this property is commonly referred to as **exact**.

We also see that if $\mathbf{G}$ has a potential, it must satisfy the property that for any indices j and k,

$$\frac{\partial G_j}{\partial x_k} - \frac{\partial G_k}{\partial x_j} = 0$$

which just follows from the equality of mixed partials. A differential form that satisfies this property is commonly referred to as **closed**. (So in $\mathbb{R}^3$, this is the same thing as having curl 0.)

So exact implies closed, and Theorem 5.62 shows us that, if our functions are defined on a convex set, closed implies exact. But closed doesn't always imply exact. The canonical example is example 1 in 5.8.

Knowing this, one thing we can ask is: for forms defined on a specific domain, can we measure how much closed forms fail to be exact? This might seem like an odd question, but the hope is that, by isolating and quantifying this specific property, we might be able to identify something intrinsically geometric about the domain. And in fact, there is a common mathematical technique that allows us to perform this "measurement".

2 Quotient Spaces

Speaking very generally, let's suppose that V is a vector space and W is some subspace. Then we'll define a new abstract vector space V/W, called the quotient space, in the following way.

Define a relation on the elements of V by saying that $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$. Then we can separate the elements of V into **equivalence classes**, where the class $[v_1]$ consists of all v_2 such that $v_2 \sim v_1$.

Probably the clearest way to think of this is that we're defining a new notion of equality on our space, where everything in the subspace W is treated as 0. So if $v_1 - v_2$ ends up in W, then it is treated as 0 under our new relation, and v_1 and v_2 are treated as equivalent.

For an example, let's take $V = \mathbb{R}^3$ and $W = \operatorname{span}\{(0,0,1)\}$, the z-axis. Then two elements (x_1, x_2, x_3) and (y_1, y_2, y_3) will be equivalent under our relation if and only if $(x_1 - y_1, x_2 - y_2, x_3 - y_3)$ lies on the z-axis; that is, if and only if $x_1 = y_1$ and $x_2 = y_2$. And the equivalence class corresponding to (x_1, x_2, x_3) will just be all elements of the form (x_1, x_2, z) for an arbitrary z. In this way, we're pretty much ignoring the third coordinate and comparing elements based only on the first two. This gives a natural way of identifying each equivalence class with an element of $\mathbb{R}^2$, given by the first two coordinates.

Now, the important thing about the quotient space construction is that we can just translate the vector space structure from V down to the set of equivalence classes V/W. We can define the sum of two equivalence classes to just be the equivalence class that contains the sum of any pair of elements from each, and we can define a scalar multiple of an equivalence class to be the class containing the scalar multiple of any element.

At this point, we do have to prove that this operation even makes sense. How do we know that, no matter which representatives we pick, their sum or scalar multiple ends up in the same equivalence class? This depends on the fact that W is a subspace.

Lemma 1. *These operations of addition and scalar multiplication are well-defined.*

Proof. Given $v_1 \sim v_2$ and $v_3 \sim v_4$, we want it to show that $v_1 + v_3 \sim v_2 + v_4$. Indeed, $v_1 + v_3 - v_2 - v_4 = (v_1 - v_2) + (v_3 - v_4)$, which is a sum of elements of W, and thus lies in W.

Similarly, if $v_1 \sim v_2$ and c is a scalar, we want that $cv_1 \sim cv_2$. Indeed, $cv_1 - cv_2 = c(v_1 - v_2)$, which is a scalar multiple of an element of W, and thus lies in W. $\square$

Returning to our example of $V = \mathbb{R}^3$, $W = \operatorname{span}\{(0,0,1)\}$, we saw above that we can identify the set V/W with $\mathbb{R}^2$. And this identification extends to the vector space structure. If we add an equivalence class containing elements of the form $(x_1, x_2, *)$ and elements of the form $(y_1, y_2, *)$, then we'll get the equivalence class containing all elements of the form $(x_1 + y_1, x_2 + y_2, *)$, just as we want.

There's another example of the quotient construction which doesn't use vector spaces, but which you may be more familiar with. Consider the integers $\mathbb{Z}$, and the subset $3\mathbb{Z}$—that is, all multiples of 3. Here, the integers have the structure of a ring: an algebraic system with operations of addition and multiplication. Though the formal particulars are slightly different, we can get a quotient ring $\mathbb{Z}/3\mathbb{Z}$, which is just the result of setting all multiples of 3 to 0, or equivalently declaring two integers to be in the same equivalence class if they differ by a multiple of 3. This system is just the integers mod 3.

The idea of a quotient comes up over and over again throughout math, and it's something you'll see in more detail if you take any abstract algebra class.

3 Measuring Inexactness, With Exercise 5.8.4

The set of 1-forms on an open set of $\mathbb{R}^n$ is a vector space (albeit an infinite-dimensional one) over $\mathbb{R}$. Also, the collections of closed and exact 1-forms are subspaces of the space of 1-forms: if we add two forms which satisfy the derivative condition to be closed, the sum will be closed by the linearity of the derivative. And if 1-forms $\mathbf{G_1}$ and $\mathbf{G_2}$ have potentials f_1 and f_2, then $f_1 + f_2$ is a potential for their sum.

So if we want to understand the difference between closedness and exactness in the space of 1-forms on a domain, it makes sense to take the quotient of the space of closed forms by the space of exact forms. When we do this, we're effectively setting every exact form equal to 0, so we're just looking at the different ways closed forms can deviate from exactness.

And, although it wasn't how the problem was phrased, in Exercise 5.8.4 you did exactly this. The end result of the problem is that for any 1-form $\mathbf{F}$ in the punctured plane $\mathbb{R} - \{(0,0)\}$, there is some multiple $(\alpha/2\pi)\mathbf{F_0}$ of the form

$$\mathbf{F_0} = \frac{-ydx + xdy}{x^2 + y^2}$$

such that $\mathbf{F} - (\alpha/2\pi)\mathbf{F_0}$ is exact. This is just the same thing as saying that, in the quotient space of closed forms by exact forms, every form is *equivalent* to some multiple of $\mathbf{F_0}$. Put another way, this quotient space is spanned by (the equivalence class of) $\mathbf{F_0}$. It's 1-dimensional!

So we now have a precise quantification of how closed forms fail to be exact in the punctured plane: it can happen, but there's only one dimension in which things can go wrong.

4 De Rham Cohomology

This is all a special case of a much more general concept known as de Rham cohomology, and using the material of section 5.9, we can go into a bit more detail about this.

In that section, we see how we can define an operation on differential forms called the **exterior derivative** d, which takes k-forms to $k+1$-forms. Using $\Omega^k(U)$ to denote the space of k-forms on a domain U, we can illustrate this like so:

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

And it turns out that the gradient, curl, and divergence are all different manifestations of this operator applied to forms on $\mathbb{R}^3$.

The exterior derivative is always a linear operator, and it has the property that applying it twice always gives 0. This is why the curl of a gradient is 0, and why the divergence of a curl is 0.

With these properties, our notions of closed and exact forms generalize: a form is closed if its exterior derivative is 0 (in the case of 1-forms, if the curl is 0) and it is exact if it is the exterior derivative of some form (in the case of 1-forms, if it has a potential function which it is the gradient of). Or in more algebraic terms: the closed forms are the kernel of the exterior derivative, and the exact forms are in the image. Since closed forms are exact, the image is contained in the kernel.

We have a sequence of things (such as the spaces of 1-forms, 2-forms,. . .) and maps from each thing to the next (such as the exterior derivatives) such that composing any two consecutive maps gives 0: the image of each one is contained in the kernel of the next one. This situation appears often in math, and it's called a **cochain complex**.

Given a cochain complex, we usually want to calculate the **cohomology**, which is the quotient of the kernel by the image at each step. In the case of the cochain complex consisting of the differential forms on some domain, its cohomology is referred to as the de Rham cohomology of that domain.

So in other, fancier-sounding words, what you showed in problem 5.8.4 was: the first de Rham cohomology of the punctured plane is $\mathbb{R}^1$.

Using tools of algebraic topology, there's a certain sense in which this tells you that the space has a single 1-dimensional "hole" (that is, the puncture at the origin). And more generally, there are connections between the de Rham cohomology — an analytic/algebraic concept — and the geometry of the space it's defined on.